REMARKS ON THE POSTAGE STAMP PROBLEM WITH APPLICATIONS TO COMPUTERS

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1 Introduction

The postage stamp problem states that an envelope may be franked with a total of at most s stamps while one has available N stamp denominations. Given s and N find the maximal integer $S = S(s,N)$ such that all integer postage values p $(1 \leq p \leq S)$ can be made up but it is not possible to build the total $S+1$. Also, all sets of N denominations that satisfy this condition are to be found.

For example, if $s = 2$ and $N = 4$ then $S(2,4) = 12$. The solution set is (uniquely) 1, 3, 5, 6 and a construction of the consecutive integers is as follows:

1	$3+1$	$6+1$	$5+5$
$1+1$	5	$5+3$	$6+5$
3	6	$6+3$	$6+6$

Clearly $S(1,N) = N$. (This means you can put only one stamp on each envelope but you have available N different denominations of stamps to chose from.) The solution set is to choose the denominations as the consecutive integers up to and including N, that is $r_i = i$ $(1 \leq i \leq N)$. It is also obviously true that $S(s,l) = s$. (This means you can put s stamps on each envelope but you have only one denomination to choose from.) In this case $r_1 = 1$.

The only other known closed form result appears in Stanton, Bate and Mullin [5] in which they establish that

$$(1) \quad S(s,2) = [(s^2 + 6s + 1)/4].$$

(Where $[x]$ represents the greatest integer less than or equal to x).

The nonsymmetry of the problem can be observed by noting that a closed form expression for $S(2,N)$ has not been obtained and appears to be a rather difficult problem.

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The problem definition is often modified so that an additional stamp of value 0 is added to the set and a letter is required to be franked with exactly s stamps.

We believe the problem has been around for a long time; however the earliest references we could find are Legard [1] and Sprague [4], in which special cases of the problem appeared.

The next section of the paper gives an application of the postage stamp problem to index registers on modern computers. Section 3 gives some new results on lower bounds for $S(s, N)$. In Section 4, the main results of Section 3 are proven. Section 5 consists of a series of tables which give all previously known special results along with some new computations of $S(s, N)$.

2 Application to Computers

It is well known that a digital computer has a memory that is a collection of cells. Each cell is named by an integer called its address. The set of all addresses are consecutive starting at zero. Instructions executed by the computer can retrieve and change the contents of a cell by specifying the address of that cell. Besides its memory, a computer has a set of registers that are also identified by integer names. The registers are used to hold operands and the results of arithmetic operations. In addition to this, the registers may be used by instructions to specify cell addresses.

The technique by which an instruction specifies an address is different on different machines, and in fact has undergone evolutionary change over the past several years. The method most used on today's large-scale computers is exemplified by the IBM 360 and 370 series of computers, in which an address is an ordered integer triple (b, i, j). The triple names the cell whose address is $b + r_i + r_j$, where r_n is zero if $n = 0$ and r_n is the contents of register n otherwise. There are restrictions on the components of the triple; namely

(2) $\quad 0 \le b < P \quad \text{and} \quad 0 \le i, j \le R.$

P, called the page size, is very much smaller than the total number of addresses available on the computer and also is much smaller than the total number of addresses referenced by a typical program. Normally, P is 2^{12}, the maximum possible address is 2^{24}, and the typical program address range is from 2^{16} to 2^{21}. For reasons of economy, the number of registers, R, is limited to 16.

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Because of these restrictions programs expend considerable amounts of their processing time calculating register contents in order to address cells. Also, many program construction techniques necessitate the ability to access large blocks of cells with contiguous addresses. (See, for example, the description of FORTRAN COMMON in Seeds [3].)

The ability to access large contiguous blocks of cells without having to dynamically compute register contents may be achieved by noting a correspondence between the postage stamp problem for $s = 2$ and the above addressing scheme. Assume that N of the R registers are available for use in address specification. Let $r_i = P \cdot u_i$ $(1 \leq i \leq N)$, where $u_1, \ldots, u_N$, is a solution set for the postage stamp problem $S(2, N)$. Also, let $u_0 = 0$, then the address (b, i, j) $(0 \leq i, j \leq N)$ is $b + r_i + r_j = b + P \cdot u_i + P \cdot u_j = b + P \cdot (u_i + u_j)$. The sum $u_i + u_j$ may achieve any value between 0 and $S(2, N)$ inclusive, because, as we have already pointed out, the equivalent postage stamp problem may be considered with $N + 1$ stamp denominations, including zero, and the requirement that a letter always be franked with exactly two stamps. Note that although $0 = 0 + 0$ is illegal postage, it is a legal address. Recall that $r_0 = 0$ when used in an address specification provides us with a free $N + l$ register for our purposes.

Since $0 \leq b < P$, it follows that all addresses from 0 through $P \cdot (S(2, N) + 1) - 1$, or a span of $P(S(2, N) + 1)$ can be achieved by the use of N registers with constant content. It is natural to inquire whether this result is optimal. The answer is not known. Restate the postage stamp problem so that, while total postages must be integers, the stamp denominations can be rational fractions. Would the value of $S(2, N)$ be greater than it is with the full integer restriction? If so, the register contents could be P times the members of a solution set of fractions with denominator P.

3 Bounds on S

It is very easy to establish a crude upper bound on $S(s, N)$, namely

$$(3) \quad S(s, N) \leq \binom{N+s}{s} - 1.$$

The above is obtained by observing that the right hand side is the number of combinations of $N + 1$ objects taken s at a time with repetitions allowed. The minus one occurs because the postage must always be positive.

The obvious method for establishing a lower bound is to generate as good a sequence as possible and call its range the lower bound. For example,

(4) $S(s,3) \;\geq\; s(s+2)$.

The above result was obtained by Stanton, Bates and Mullin [5]. In order to verify it simply select stamps of denominations 1, $s+1$, $s+2$. This is a good bound for small values of s but it becomes increasingly poor as s becomes large.

Wegner and Doig [6] give a generalized sequence of integers, that in fact are symmetric, and establish that

(5) $S(2,N) \;\geq 2N(j+1)-4j^2, \qquad N \geq 4j-2$

In particular,

(6) $S(2,N) \;\geq\; 4N-4$.

(7) $S(2,N) \;\geq\; 6N-16, \qquad N \geq 6$.

(8) $S(2,N) \;\geq\; 8N-36, \qquad N \geq 10$.

Results (6) and (7) were also obtained by Stanton, Bates and Mullin [5].

A constructive procedure is provided by Wegner and Doig [6] for finding sequences that generalize result (5) for $s > 2$. An improvement on these results will be presented in this paper.

By using (4) the following bound is easily obtained.

Lemma 1. $S(s,4) \geq 2s^2+s-2$.

Proof. Simply select stamps of denominations 1, s, $s+1$, s. By result (4), 1, s, $s+1$ yields all denominations up to s^2-1 using up to $s-1$ stamps of denominations 1, s and $s+1$ only. It is straightforward to check that all denominations up to $2s^2+s-2$ are achievable.

In order to improve on the lower bound for $S(2,N)$ let $N = 4k$ and consider the following sequence.

$$(9) \qquad r_i \;=\; \begin{cases} i & \text{for } 1 \leq i \leq k. \\ i(k+1)-k^2 & \text{for } k < i \leq 3k. \\ 2k^2+i & \text{for } 3k \leq i \leq 4k. \end{cases}$$

For example, if $N = 8$, the sequence becomes

1, 2, 5, 8, 11, 14, 15, 16.

With this sequence all postage stamp denominations up to $(N^2+8N)/4$ can be reached by putting at most two stamps on the envelope. By a slight modification of (9) this result can be extended to all N.

Lemma 2. If $N = 4k + r$, $0 \le r \le 3$ then

$$(10)\quad S(2,N) \;\ge\; (N^2 + 8N - r^2)/4.$$

Proof. It is straightforward to verify that the following sequence of denominations will yield the desired result.

$$(11)\quad r_i \;=\; \begin{cases} i & \text{for } 1 \le i \le k. \\ i(k+1) - k^2 & \text{for } k < i \le 3k + r. \\ 2k^2 + i + rk & \text{for } 3k + r < i \le 4k + r. \end{cases}$$

Results similar to the above can be obtained for $s > 2$. This is done constructively in the following theorem, by determining a lower bound on $S(s,N)$.

Theorem 1. If $N = qs + r$ where $0 \le r < s$, then

$$(12)\quad S(s,N) \;\ge\; \sum_{i=1}^{s} \binom{s+i}{2i} q^i + r \cdot \sum_{i=0}^{s-1} \binom{s+i-1}{2i} q^i.$$

Proof. By construction of an appropriate sequence. For the details see the next section.

Corollary. If $N = s$ then (12) becomes

$$(13)\quad S(s,s) \;=\; f_{2s+1} - 1$$

where f_i is the i$^{\text{th}}$ fibonacci number.

Proof. If $N = s$ then $q = 1$, $r = 0$ and (12) becomes

$$\begin{aligned} S(s,s) \;&=\ge\; \sum_{i=1}^{s} \binom{s+i}{2i} \\ &=\; (1/\sqrt{5})\left(\left(\frac{\sqrt{5}+1}{2}\right)^{2s+1} + \left(\frac{\sqrt{5}-1}{2}\right)^{2s+1}\right) - 1 \end{aligned}$$

Q.E.D

Theorem 1 is really only useful when $N \ge s$. It is too weak when $s > N$ for then $q = 0$. The next theorem gives a lower bound on $S(s,N)$, in terms of smaller values of s and N.

Theorem 2. If $k \ge 0$ then

$$(14)\quad S(s+k, N+k) \;\ge\; 2^k(S(s,N) + s + 2) - s - k - 2.$$

Proof. By construction. For the details see the next section.

4 Proofs of Theorem 1 and 2

Proof of Theorem 1. If $N = qs + r$, $0 \le r < s$, a denomination set can be constructed for each s and N in which the set of N denominations consists of $s - 1$ groups of q denominations and one group of $q + r$ denominations. Within each group the denominations form a linear sequence. For example, if $s = 3$ and $N = 13$ (then $q = 4$ and $r = 1$) the denomination set is

1	2	3	4	
9	14	19	24	
53	82	111	140	169

and it follows that $(3, 13) \ge 197$.

To construct the lower bound we will first consider the case when $r = 0$. Hence we must construct s groups of q denominations each. The construction of each successive group of q denominations depends on the previous groups. The first group consists of the consecutive integers $1, 2, \ldots, q$. In order to simplify the description of the construction of the remaining $s - 1$ groups we introduce the following notation.

$F(i)$ is the first member (i.e. smallest denomination) in group i.

$L(i)$ the last member (i.e. largest denomination) in group i.

$D(i)$ is the difference between consecutive denominations in group i.

$B(i)$ is the span of the first i groups using no more than i stamps to frank a letter.

$B'(i)$ is the span of the first i groups using no more than $i + 1$ stamps to frank a letter.

Clearly,

$$F(1) = 1,\ L(1) = q,\ D(1) = 1,\ B(1) = q,\ B'(1) = 2q.$$

and

$$(15)\quad L(i) \quad = \quad F(i) + (q - 1)D(i).$$

To construct the sequence recursively, let

$$(16)\quad F(i + 1) = B'(i) + 1 \quad \text{and} \quad D(i + 1) = B(i) + 1.$$

From (15) and (16), and a counting argument it can be shown that

(17) $B(i+1) \;=\; L(i+1)+B(i).$

Also,

(18) $B'(i+1) \;=\; L(i+1)+B(i+1).$

To see (18) first note that all sums not greater than $L(i+1)$ can be made up using no more than $i+1$ denominations since $L(i+1 < B(i+1)$. All sums from $L(i+1)+1$ through $B(i+1)+L(i+1)$ can be made using the single denomination $L(i+1)$ and no more than $i+1$ stamps to form any total between 1 and $B(i+1)$. Thus with no more than $i+2$ stamps each sum from 1 through $B(i+1)+L(i+1)$ can be achieved.

Using (16), (18), and (17) respectively it follows that

(19) $F(i+1) \;=\; 2B(i)-B(i-1)+1.$

Using (17), (15), (19), and (16) respectively a recurrence relation for $B(i)$ is obtained. Namely,

(20) $B(i+1) \;=\; (q+2)B(i)-B(i-l)+q.$

Since $B(0)=0$ and $B(1)=q$, (20) can be solved to yield

(21) $$B(i) \;=\; \sum_{j=1}^{i}\binom{i+j}{2j}q^j.$$

Since $S(s,N)\geq B(s)=\sum_{j=1}^{s}\binom{s+j}{2j}q^j$, this completes the proof for the case where $r=0$.

When $0<r<q$ we can add to (21) the r denominations of the form

(22) $L(s)+kD(s), \qquad 0<k\leq r.$

Using an argument similar to the case $r=0$, it can be shown that the span of the augmented set is $B(s)+rD(s)$, thus

(12) $$S(s,N) \;=\; \sum_{i=1}^{s}\binom{s+i}{2i}q^i + r\sum_{i=0}^{s-1}\binom{s+i-1}{2i}q^i. \qquad \text{Q.E.D.}$$

Proof of Theorem 2. For a fixed s and N let $B(k)$ be the lower bound for $S(s+k,N+k)$. Clearly, $B(0)=S(s,N)$. To generate the value of

$B(k+1)$ append a denomination of $B(k)+1$ to the existing set of $k+s$ denominations. Then, all values from 1 through $B(k)$ can be generated using the original $k+s$ denominations, $B(k)+1$ is generated with the one new denomination, all values of $B(k)+2$ through $2B(k)+1$ are generated using the new denomination along with $k+s$ or less of the original denominations, $2B(k)+2$ is obtained by using the new denomination twice, and $2B(k)+i$ $(3 \leq i \leq k+s+1)$ is generated using $B(k)+1$ twice along with 1, $i-2$ times. Thus

(23) $B(k+1) \;=\; 2B(k)+s+k+1.$

The above can be solved to yield

(24) $B(k) \;=\; 2^k(S(s,N)+s+2)-s-k-2$

This completes the proof of Theorem 2. Q.E.D.

5 Tables

Extensive computations have been made on the postage stamp problem, primarily by Lunnon [2] and Stanton, Bate and Mullin [5]. These computations are combined and updated in the following tables. In particular, we have recently computed that

(25) $S(2,13) \;=\; 72,$

and its unique solution set. One quickly observes in the tables that the solution sets for a given N are not always unique. Due to the way that the integers combine the problem of determining whether or not a given solution set is unique appears to be a difficult problem.

In order to extend Table 2 (i.e. compute $S(2,14)$) via the algorithms by which (25) was computed we estimate that it would take greater than 40 hours on an IBM 370 mod 158 (Mod III). However we can make the observation that by choosing stamps of denominations 1, 3, 4, 5, 8, 14, 20, 26, 32, 35, 36, 37, 39, 40 it follows that $S(2,14) \geq 80$. Also Sprague [4] gives the sequence 1,3, 4, 9, 11, 16, 20, 25, 30, 34, 39, 41, 46, 47, 49, 50 thus establishing that $S(2,16) \geq 100$.

In Table 1 all the known values, excluding the trivial cases, of $S(s,N)$ are given. Tables 2 through 15 give all the known solution sets for the respective postage stamp problem. One quickly observes that the solution sets for a given s and N are not always unique.

$s\backslash N$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	3	4	5	6	7	8	9	10	11	12	13
2	2	4	8	12	16	20	26	32	40	46	54	64	72
3	3	7	15	24	36	52	70	93	121				
4	4	10	26	44	70	108	162						
5	5	14	35	71	126	211							
6	6	18	52	114	216	388							
7	7	23	69	165	345								
8	8	28	89	234	512								
9	9	34	112	326	797								
10	10	40	146	427									
11	11	47	172	547									
12	12	54	212	708									
13	13	62	259	873									
14	14	70	302	1094									
15	15	79	354										
16	16	88	418										

Table 1: $S(s,N)$

N	$S(2,N)$	Solution Sets
1	2	(1)
2	4	$(1,2),(1,3)$
3	8	$(1,3,4)$
4	12	$(1,3,5,6)$
5	16	$(1,3,5,7,8)$
6	20	$(1,2,5,8,9,10),(1,3,4,8,9,11),(1,3,4,9,11,16),$
		$(1,3,5,6,13,14),(1,3,5,7,9,10)$
7	26	$(1,2,5,8,11,12,13),(1,3,4,9,10,12,13),(1,3,5,7,8,17,18)$
8	32	$(1,25,8,11,14,15,16),(1,3,5,7,9,10,21,22)$
9	40	$(1,3,4,9,11,16,17,19,20)$
10	46	$(1,2,3,7,11,15,19,21,22,24),(1,2,5,7,11,15,19,21,22,24)$
11	54	$(1,2,3,7,11,15,19,23,25,26,28),(1,2,5,7,11,15,19,23,25,26,28),$
		$(1,3,4,9,11,16,18,23,24,26,27),(1,3,5,6,13,14,21,22,24,26,27)$
12	64	$(1,3,4,9,11,16,21,23,28,29,31,32)$
13	72	$(1,3,4,9,11,16,20,25,27,32,33,35,36)$

Table 2:

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N	$S(3,N)$	Solution Sets
1	3	(1)
2	7	$(1,3)$
3	15	$(1,4,5)$
4	24	$(1,4,7,8)$
5	36	$(1,4,6,14,15)$
6	52	$(1,3,7,9,19,24), (1,4,6,14,17,29)$
7	70	$(1,4,5,15,18,27,34)$
8	93	$(1,3,6,10,24,26,39,41)$
9	122	$(1,3,8,9,14,32,36,51,53)$

Table 3:

N	$S(4,N)$	Solution Sets
1	4	(1)
2	10	$(1,3), (1,4)$
3	26	$(1,5,8)$
4	44	$(1,3,11,18)$
5	70	$(1,3,11,15,32)$
6	108	$(1,4,9,16,38,49), (1,5,8,27,29,44)$
7	162	$(1,4,9,24,35,49,51), (1,4,10,15,37,50,71),$
		$(1,5,8,25,31,52,71)$

Table 4:

N	$S(5,N)$	Solution Sets
1	5	(1)
2	14	$(1,4)$
3	35	$(1,6,7)$
4	71	$(1,4,12,21), (1,5,12,28)$
5	126	$(1,4,9,31,51)$
6	211	$(1,4,13,24,56,61), (1,5,8,33,54,67)$

Table 5:

N	$S(6,N)$	Solution Sets
1	6	(1)
2	18	$(1,4),(1,5)$
3	52	$(1,7,12)$
4	114	$(1,4,19,33)$
5	216	$(1,7,12,43,52)$
6	388	$(1,7,11,48,83,115)$

Table 6:

N	$S(7,N)$	Solution Sets
1	7	(1)
2	23	$(1,5)$
3	69	$(1,8,13)$
4	165	$(1,5,24,37)$
5	345	$(1,8,11,64,102)$

Table 7:

N	$S(8,N)$	Solution Sets
1	8	(1)
2	28	$(1,5),(1,6)$
3	89	$(1,9,14)$
4	234	$(1,6,25,65)$
5	512	$(1,9,15,78,115),(1,9,15,80,118)$

Table 8:

N	$S(9,N)$	Solution Sets
1	9	(1)
2	34	$(1,6)$
3	112	$(1,9,20)$
4	326	$(1,5,34,60)$
5	797	$(1,9,23,108,181)$

Table 9:

N	$S(10,N)$	Solution Sets
1	10	(1)
2	40	$(1,6),(1,7)$
3	146	$(1,10,26)$
4	427	$(1,6,41,67)$

Table 10:

N	$S(11,N)$	Solution Sets
1	11	(1)
2	47	$(1,7)$
3	172	$(1,9,30),(1,10,26)$
4	547	$(1,7,48,85)$

Table 11:

N	$S(12,N)$	Solution Sets
1	12	(1)
2	54	$(1,7),(1,8)$
3	212	$(1,11,37)$
4	708	$(1,7,48,126)$

Table 12:

N	$S(13,N)$	Solution Sets
1	13	(1)
2	62	$(1,8)$
3	259	$(1,13,34)$
4	873	$(1,9,56,155)$

Table 13:

N	$S(14,N)$	Solution Sets
1	14	(1)
2	70	$(1,8),(1,9)$
3	302	$(1,12,52)$
4	1094	$(1,8,61,164)$

Table 14:

N	$S(15, N)$	Solution Sets
1	15	(1)
2	79	$(1, 9)$
3	354	$(1, 12, 52)$

Table 15:

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