Inequalities

Jeffrey A. Barnett
jbb@notatt.com

Problems

Prove the following inequalities, $x \geq 0$. Better, find a common proof for all of them. Solutions follow.

1. $\frac{1}{a^2}(e^{ax}-1)(1-e^{-ax}) \geq x^2 \qquad a \neq 0$
2. $(ax + x^2/2)\log\left(\frac{a+x}{a}\right) \geq x^2 \qquad a > 0$
3. $(ax - x^2/2)\log\left(\frac{a}{a-x}\right) \geq x^2 \qquad 0 \leq x < a$
4. $\log\left(\frac{\cos(a)}{\cos(a+x)}\right) \cdot \log\left(\frac{\sin(a+x)}{\sin(a)}\right) \geq x^2 \qquad 0 < a, \; 0 \leq x < \frac{\pi}{2} - a$
5. $\frac{\log\log(e+x)}{4}[2(e+x)^2\log(e+x) - x^2 - 2ex - 2e^2] \geq x^2 \qquad e = \exp(1)$
6. $\log\left(\frac{1+e^x}{2}\right)(x - e^{-x} + 1) \geq x^2 \qquad e = \exp(1)$
7. $\arctan(x)(x + \frac{x^3}{3}) \geq x^2$
8. $\frac{1}{2}\log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) \times \left[x + \sqrt{x^2+a^2} + a^2\log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)\right] \geq x^2 \qquad a > 0$
9. $\frac{4}{3}\Big((1+x)^2 - (1+x)^{3/2} - (1+x)^{1/2} + 1\Big) \geq x^2$

Solutions

Let $v(t)$ be a strictly positive function and define

$$E(x) = \int_0^x v(t)\, dt \qquad F(x) = \int_0^x v(t)^{-1} dt.$$

Then it is easy to show that

$$\min_v E(x)F(x) \quad = \quad x^2$$

and that minimum is achieved for $v(t) = c$, where c is an arbitrary positive constant. Thus, $E(x)F(x) \geq x^2$ for any nonconstant choice of of positive v. The above inequalities are demonstrated by making the following choices for the function, v:

1. $v(x) = e^{ax}$
2. $v(x) = a + x$
3. $v(x) = a - x$
4. $v(x) = \tan(a + x)$
5. $v(x) = (e + x)\log(e + x)$
6. $v(x) = \dfrac{e^x}{1 + e^x}$
7. $v(x) = 1 + x^2$
8. $v(x) = \sqrt{x^2 + a^2}$
9. $v(x) = \sqrt{1 + x}$