Calibrated Critics

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Abstract

Critics are simply functions that return values that can be compared in order to determine preferences between domain elements. Therefore, the definition of a critic will include a requirement that the function's range support a useful order relation. The main goal herein is to investigate when two critic must be identical, i.e., agree on the value of every element in their joint domain. One restriction investigated is that two critics must be in *general agreement*: That simply means there will never be a case where one critic strongly prefers one element to another and the other has a strong preference the other way around. The other restriction considered is that critics be *calibrated*: That means for every value, r, in their joint range, there will exist an element in their joint domain such that both rate that element r.

Requiring critiques to be in general agreement and calibrated does not imply they are identical (the same function). However, it is enough to show that, while not identical, they are the "next best thing." If a further assumption is made about their range, identity will be assured.

1 Definitions and Other Preliminaries

Critics, in the real world, rate things to reflect how well they like them. Ratings typically reveal some inner preference ranking and we assume that ranking is compatible with a *strictly linear order*. If this article were about psychology and preference relations, this assumption would not be valid; in particular humans often hold non-transitive preference relations.[1]

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[1]Barnett, J.A. "Preference relations." `http://notatt.com/preferences.pdf`

Definition. A *Strictly Linear Order*, $\succ$, is antireflexive ($x \not\succ x$), antisymmetric ($x \succ y \to y \not\succ x$), but is transitive ($x \succ y \land y \succ z \to x \succ z$), and obeys trichotomy (for any x and y, either $x \succ y$, $y \succ x$, or $x = y$). The operators $\preceq$, $\prec$, and $\succeq$ are derived from $\succ$ in the usual way. □

We can now define a critic:

Definition. A *Critic*, c, is a total function from a set of elements, X, to an ordered range, $(R, \succ)$, where R is the set of possible ratings, $\succ$ is a strict linear order on R, and $r_1 \succ r_2$ means that objects rated r_1 are preferred to those rated r_2. The critic is onto R so that $c(X) = R$, □

Definition. The critics, c_1 and c_2, are *in the same business* if their domains are the same and their ranges, including order operators, are the same. □

A critic, as defined here, is just an arbitrary total function with an ordered range. This definition might be more general than expected.

2 Critic Pairs

As our objective is to see whether two critics that obey certain restrictions are identical, we make the natural and reasonable assumption, throughout, that c_1 and c_2 are in the same business.

Notation. Below, critics c_1 and c_2 are always assumed to be in the same business. Their common domain is X; x and x_i, where $i \geq 1$, are always elements of X. Their common range is $(R, \succ)$; r and r_i, where $i \geq 1$, are always elements of R and $\succ$ is a strictly linear order. □

Next, two restrictions—general agreement and calibration—on critics c_1 and c_2 in the same business are introduced.

Definition. The critics, c_1 and c_2, are said to be in *general agreement* if, for all x_1 and x_2 in X,

$$\big(c_1(x_1) \succ c_1(x_2) \to c_2(x_1) \succeq c_2(x_2)\big) \land \big(c_2(x_1) \succ c_2(x_2) \to c_1(x_1) \succeq c_1(x_2)\big).$$

If a x_1 and x_2 pair do not obey the above, they are *dissident*. □

Ratings in R, so far, have no inherent meaning other than with reference to the order relation. The next restriction, assumed below on a pair of critics, is an attempt to give some basis for what a rating means by *requiring examples of common usage.*

Definition. The critiques, c_1 and c_2, are said to be *calibrated* if for every r in their common range, there exists a x in their common domain such that $c_1(x) = c_2(x) = r$. □

3 A Redefined Domain

An equivalence relation on X simplifies our investigation.

Definition. Two elements, x_1 and x_2 in X, are *equivalent*, written $x_1 \doteq x_2$, if and only if $c_1(x_1) = c_1(x_2)$ and $c_2(x_1) = c_2(x_2)$. □

Notation. Θ is the set of partitions induced by "$\doteq$" on X. A particular element may be cited as $\theta_{r_1 r_2}$, where $c_1(\theta_{r_1 r_2}) = r_1$ and $c_2(\theta_{r_1 r_2}) = r_2$. Several abbreviations will be used: $\theta_{r_i r_j}$ as θ_{ij}; $\theta_{r_i r_i}$ as θ_i; θ_{rr} as θ_r; etc. If a partition is empty, a reference to it is undefined. In obvious contexts, *this notation simply denotes a representative element of the partition.* □

Observation. c_1 and c_2 are *calibrated critics* if and only if θ_r is defined for all r in R. Two critics are *identical* if and only if they are calibrated and $\theta_{r_1 r_2}$ is undefined when $r_1 \neq r_2$. Thus, we know everything about the sameness of two critics if we know which classes have elements and which are empty! □

Notation. Therefore, *we continue our discussion assuming the domain of the critics is just these nonempty equivalence classes or, more accurately, a representative of each nonempty class*; in particular, the domain is now Θ, rather than X. □

Observation. c_1 and c_2 are *not* in general agreement if there exist dissident θ_{ij} and θ_{mn} in Θ; otherwise c_1 and c_2 are in general agreement. That θ_{ij} and θ_{mn} are dissident means that $r_i \succ r_m$ while $r_n \succ r_j$ or the other way round; in either case, the requirement of general agreement is violated. □

4 More on Orders

It's almost time to derive our main results but first some terminology about orders is needed.

Definition. Let r_1 and r_2 be elements of R. We say that r_1 and r_2 are *adjacent in* R or simply *adjacent* if there does not exist a r in R such that $r_1 \succ r \succ r_2$ or $r_2 \succ r \succ r_1$. □

Definition. Let $\succ$ be a strict linear order on R. If there are no adjacent elements in R, we will say that $\succ$ *is a dense ordering on* R, i.e, if $r_1 \succ r_2$ in R, there exists a r in R such that $r_1 \succ r \succ r_2$. □

Observation. The astute reader may ask why the type of order relation required is not a well ordering. A *well ordering* specifies that a least (or greatest) element can be found in any subset of its domain. Don't we want to ask for a critic's favorite or favorites among any set of choices it is given? If the ordered set is finite, it is well ordered; if it is infinite, it may or may not be with the given order relation. We will dabble with infinite domains below and do not want to preclude using familiar order operators that are not well orderings, e.g., ">" applied to the reals. □

5 Critic Equality

We now have all the terminology and observations necessary to draw simple conclusions about critic equality. We start with a repeat from above.

Result. The critics c_1 and c_2 in the same business are calibrated if and only if θ_r is defined for every r in R. □

Result. Assume c_1 and c_2 are in the same business, calibrated, and in general agreement: Then neither θ_{12} nor θ_{21} is defined when r_1 and r_2 in R are neither equal nor adjacent. Say that $r_1 \succ r_2$, then nonadjacent values have an interloper; there exists a r in R such that $r_1 \succ r \succ r_2$. The existence of θ_r, guaranteed by calibration, is incompatible with general agreement. □

Result. If r_1 and r_2 are adjacent then at most one of θ_{12} and θ_{21} may be defined. There are no other circumstances where a θ_{ij}, $i \neq j$, is defined. □

Result. If $\succ$ is a dense ordering on R, the only defined domain elements are θ_r for each r in R, because there are no adjacent ratings. In other words, c_1 and c_2 are identical. □

6 After Thoughts

This section relates the original problem that motivated this article and the inspiration for the name, *Calibrated Critics*.

Problem Origin

When I originally thought about this problem many years ago, I start from a quite different point from this article. I asked myself a question that I found rather fascinating:

> Assume that (partially) derivatives of two functions agreed as to increasing, decreasing, or zero at each domain element (and direction), what other assumptions would be necessary to conclude the functions were identical?

I found answering this question very difficult and still do. The project covered herein resulted from attempts to simplify the problem. A solution of the original problem might be something along these lines:

> Two total differential functions with the same domain onto the same range are identical if their (partial) derivatives agree in sign at each domain point (and direction) and, for each value in their joint range, there is a point in their domain where both functions achieve that value. Note that the domain is a subset of $\Re^n$, the range is a subset of $\Re$, and the standard numerical "$>$" is the range order relation.

The thing to note here is that the role of "general agreement" is played by a local property: sign of a derivative. In the problem discussed in this note, general agreement is a global property. So the open question is whether there are additional simple assumptions that would entail the desired result.

The Name of the Thing

"Calibrated Crickets" was the name of an article I read many years ago in the *Worm Runner's Digest* (W.R.D.) and it suggested "Calibrated Critics" as the name to this article. Since our discussion is completely off topic now, I'll provide some information on the W.R.D. and crickets.

In the 1950s an interesting line of research involving planarian worms was initiated. Some worms were "taught" or "discovered" how to navigate a maze. These students were prepared, feed to a new generation of worms, and the maze learning of these new ones was measured and compared to the old. A researcher in the area first published the *Journal of Neuropsychiatry* then in 1966 changed the name to *Journal of Biological Psychology*.

In either case you could do the following: hold a copy of the Journal with the front cover facing you; place hands in the middle of the left and right sides; rotate the Journal 180° around the axis between your two hands; you are now staring at the front cover of the *Worm Runner's Digest* of course the two(?) journals are printed upside down to one another.

The W.R.D. was devoted exclusively to satirical articles. One that I had the good fortune to read was entitled *Calibrated Crickets*. The problem it purported to solve was exact temperature control when perishable goods were shipped. The solution ran as follows: take crickets, one at a time, and put them on a metal strip that was secured to an ice block on one end and a hot iron on the other. The temperature will vary continuously between the two extremes and the cricket will move to the place on the strip that suits it best. Note the temperature at that spot and write it on the cricket's back with a soft tipped pen. The cricket is now calibrated.

When a shipment is ready for transport, select a cricket calibrated to the preferred temperature during shipment. The cricket's chirp rate will measure the difference between the cricket's preferred temperature and the ambient. The control logic uses the chirping as feedback to control the heating and cooling mechanisms. One could easily tell this article was a satire and not serious; after all there was no cost or accuracy comparison between crickets on one hand and thermocouples on the other.

The journal, both sides, ceased publication in 1979 and I could not locate an archive or a source for either purchase or search. Thus, I can not provide a proper citation for this inspirational article.