The Number of Almost Linear Sequences[1]

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Problem: Let $\mathcal{J}$ be a class of integer-valued functions defined on $\mathbf{N}^+$, the positive integers. If $j \in \mathcal{J}$, then $j(1) = 1$ and

$$j(m) + j(n) - 1 \leq j(m+n) \leq j(m) + j(n)$$

for all $m, n \in \mathbf{N}^+$. Show that the number of distinct initial sequences of length N generated by the $j \in \mathcal{J}$ is

$$\sum_{1 \leq n \leq N} \varphi(n),$$

where $\varphi(n)$ is Euler's Totient function, i.e., $\varphi(n)$ is the number of positive integers that are less than or equal to n and relatively prime to n.

Solution: Every $j \in \mathcal{J}$ has one of the two forms, either

1. $j(n) = \lfloor rn + 1 \rfloor$, for a $0 \leq r < 1$, or
2. $j(n) = \lfloor kn/p + 1 - 1/p \rfloor$, for relatively prime integers, $0 < k \leq p$.

An initial sequence generated by the first form, when r is irrational, is also generated by some $k/p \approx r$ where k and p are relatively prime integers. Thus, if only finitely many terms are viewed, 1 can be rewritten as

1. $j(n) = \lfloor kn/p + 1 \rfloor$, for relatively prime integers, $k < p$,

plus the case where $j(n) = 1$ for all n.

Therefore, for every relatively prime $k < p$ there are a pair of functions $j_1, j_2 \in \mathcal{J}$, where $j_1(n) = \lfloor kn/p + 1 - 1/p \rfloor$ and $j_2(n) = \lfloor kn/p + 1 \rfloor$. Note that $j_1(n) = j_2(n)$ for all $n < p$ but $j_1(p) \neq j_2(p)$. In other words, a sequence generated by k and p bifurcates into distinct sequences at length p. In fact, there is such a bifurcation for every k relatively prime to p. Since there are $\varphi(p)$ sequence splits at length p, the claim follows.

[1] *American Mathematics Monthly Problem Section* 1993. A better solution solution is in the 1996 Monthly.
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