MONOTONIC SET FUNCTIONS

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Problem: A real-valued f defined on non-empty subsets of $\Re$ is a *monotonic set function* if $f(S \cup \{x\}) \leq f(S \cup \{y\})$ for all $S \subset \Re$ and $x \leq y$. Questions: (1) Categorize monotonic set functions when their domain is restricted to finite non-empty subsets of $\Re$, (2) Categorize the class of all monotonic set functions.

Solution: It is easy to verify that f is a monotonic set function on the domain of finite subsets of $\Re$ if and only if there is a function g such that $f(S) = g(\min(S), \max(S))$, where g is monotone non-decreasing in both of its arguments.

The general characterization rests on the observation that the definition of a monotonic set function places no constraints on the relative values of $f(S)$ and $f(T)$ unless $S - T$, the symmetric difference of S and T, is finite. Therefore, the first step is to partition $2^{\Re}$ into classes such that S and T are in the same class if and only if $S - T$ is finite. Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be such a partition. An f_λ, a monotonic set function with domain R_λ, can be chosen independently for each $\lambda \in \Lambda$. So define $f(S) = f_\lambda(S)$ when $S \in R_\lambda$.

The possible choices for each f_λ depend on characteristics of R_λ. If elements of R_λ are neither bounded above or below, then $f_\lambda(S) = c_\lambda$. If elements of R_λ are bounded above but not below, there are two cases: (1) $\sup(S) \in S$ for all $S \in R_\lambda$ and (2) otherwise. In the first case, $f_\lambda(S) = g_\lambda(\max(S))$, where g_λ is any monotone non-decreasing function. In the second case,

$$f_\lambda(S) \quad = \quad \begin{cases} c_\lambda & \text{if } s < S^* \text{ for all } s \in S; \\ g_\lambda(\max(S)) & \text{otherwise,} \end{cases}$$

where g_λ is a monotone non-decreasing function such that $g_\lambda(S^*) \geq c_\lambda$ and S^* is the greatest limit point of S. The characterization when R_λ is bounded below but not above is similar, and the case where R_λ is bounded combines the previous results.

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