Preference Relations

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1 Summary

The concept of preference is rich both mathematically and psychologically. Preferences are the choices made when there are alternative possibilities and having a preference simply means that the same choice will be made given the same set of alternatives in the same situation. Thus, preferences summarizes how, but not necessarily why, we make the choices that we do.

Preferences may be context dependent in the sense that different choices are possible from the same set of alternatives in different circumstances. However, it will be assumed herein that situation is part of the problem statement so that preferences are context free. Considerations of preference variations within changing context structures would be an interesting generalization of the ensuing discussion.

Preference is formally defined below as a choice function and the concept is related to transitivity: a transitive preference is one that is derived from an intrinsic ordering of its domain. Several properties that are mathematically equivalent to transitivity are examined as evidence that preferences should be transitive. Though that evidence may seem compelling and many would claim that an intransitive preference relation would be irrational, I will demur.

2 Preference and Transitivity

A preference is a function that makes a consistent choice among alternatives. That thought is captured by this simple definition:

Definition 1 (Preference) *Let $\mathcal{U}$ be a finite universe of elements. Then $P\colon 2^{\mathcal{U}} \to \mathcal{U}$ is a* preference *if $P(S) \in S$ for all nonempty $S \subset \mathcal{U}$.*

A preference is just a choice function defined on a finite domain: given a set of options $S \subset \mathcal{U}$, the preference (or choice) is $P(S)$. This definition precludes indifference in the sense that two choices cannot be treated as equals—one of the elements will be preferred to the other. Therefore, a preference induces an associated binary relation and an ordering of its domain:

Definition 2 (Associate binary relation) *Let $P\colon 2^{\mathcal{U}} \to \mathcal{U}$ be a preference. Then the* associate binary relation *is defined as $u\,P\,v \equiv u = P(\{u\,v\})$ for all $u, v \in \mathcal{U}$.*

This relation is reflexive ($u\,P\,u$ for all $u \in \mathcal{U}$) and antisymmetric ($u\,P\,v$ iff not $v\,P\,u$ for all $u \neq v \in \mathcal{U}$). Therefore, the well-known tournament theorem is applicable to associate binary relations because antisymmetry implies the theorem's antecedent.

Theorem 3 (Tournament theorem) *If P is a binary relation such that either $u\,P\,v$ or $v\,P\,u$ for all $u, v \in \mathcal{U}$, then there is an ordering of $\mathcal{U}$, say $u_1, \ldots, u_n$, such that $u_i\,P\,u_{i+1}$, where $1 \leq i < n$ and $n = |\mathcal{U}|$. The ordering is, in general, not unique.*

It is instructive to compare transitive relations to the binary relations associated with preferences. The definition of transitivity and the relevant ordering theorem are as follows.

Definition 4 (Transitive relation) *The binary relation R is* transitive *on a domain $\mathcal{U}$ if $a\,R\,b \wedge b\,R\,c \to a\,R\,c$ for all $a, b, c \in \mathcal{U}$.*

Theorem 5 (Transitive ordering) *If R is a transitive relation on the finite domain $\mathcal{U}$, then there is a ordering, $u_1, \ldots, u_n$ of $\mathcal{U}$, such that $u_i\,R\,u_j$ for all $1 \leq i < j \leq n$, where $n = |\mathcal{U}|$. If a transitive relation is antisymmetric, the order is unique; If only $a\,R\,b \vee b\,R\,a$ is required for all $a, b \in \mathcal{U}$, the order is not necessarily unique.*

The tournament theorem guarantees that there is an order where the first of each adjacent pair is preferred while transitivity guarantees more: each element in the order is preferred to all elements that follow it. Both ordering theorems only apply to binary relations. The next definition is the natural generalization for preferences.

Definition 6 (Transitive preference) *The preference P on $\mathcal{U}$ is a* transitive preference *if there is an ordering, $u_1, \ldots, u_n$ of $\mathcal{U}$, where $n = |\mathcal{U}|$, such that $P(S)$ is the least element of S in the given order for all nonempty $S \subset \mathcal{U}$.*

Thus, a transitive preference is completely determined by the given order. It is clearly a preference because $P(S) \in S$ for all nonempty $S \subset \mathcal{U}$ and the associated binary relation is clearly transitive and unique.

3 Restricted Preferences

Three reasonable-sounding restrictions on preferences are introduced next. It will be shown that they are all equivalent to requiring that a preference be transitive and, therefore, that the restrictions are equivalent. The following section argues that not all rational preferences are transitive in spite of the evidence to the contrary offered by the three definitions.

Definition 7 (Decomposability) *A preference P is* decomposable *if for all nonempty $S, T \subset \mathcal{U}$, either $P(S \cup T) = P(S)$ or $P(S \cup T) = P(T)$.*

Definition 8 (Extendability) *A preference P is* extendable *if for all nonempty $S \cup T \subset \mathcal{U}$, either $P(S \cup T) = P(S)$ or $P(S \cup T) \notin S$.*

Definition 9 (Stability) *A preference P is* stable *if $P(T) \in S \rightarrow P(S) = P(T)$, for all nonempty $S \subset T \subset \mathcal{U}$.*

Essentially, decomposability means that if a set of choices is split in parts, the preference of the aggregate will be the preference of one of the parts, i.e., if nonempty $S_1, \ldots, S_m \subset \mathcal{U}$,

$$P(\bigcup_{1 \leq i \leq m} S_i) \quad \in \quad \bigcup_{1 \leq i \leq m} \{P(S_i)\}.$$

The meaning of extendability is that, if an element is preferred, then additional choices will not make one of the original elements preferred when it was not previously. Finally, stability entails that removal of non-preferred elements from a set does not change the preference. The relationship between these definitions and transitivity is described by the next theorem.

Theorem 10 (Equivalence) *A transitive preference is also decomposable, extendable, and stable. A decomposable, extendable, or stable preference is also transitive.*

The proof of this simple theorem is relegated to the appendix where it is shown that transitivity, decomposability, extendability, and stability are equivalent requirements for a preference.

4 DISCUSSION

Theorem 10 states that a nontransitive preference is also nondecomposable, nonextendable, and unstable. Thus, it must surely appear that any rational being would only employ transitive preferences. This is an obvious and appealing conclusion. Perhaps it says something about me that I don't believe a word of it.

I hope an example will help salvage my reputation: I know a women who has given a lot of thought to how she chooses an escort when she has options. It turns out that she equally values three things in her dates—charm, looks, and intelligence. The fourth thing she cares about is taste in movies, but she doesn't count this as highly as the other three criteria.

Currently, there are three men who call her frequently enough that she often must make a choice. Their names are Ron, Sam, and Tom. As I am told, her rankings, high to low, in terms of charm are Ron, Sam, and Tom. In regards to looks it's Tom, Ron, and Sam, while in terms of intelligence it's Sam, Tom, and Ron. For movies, she likes the choices made by Tom, Sam, and Ron in that order.

Now it is easy to understand how she chooses a date—she selects the man who is better in more of the three ways she cares most about. If there is a tie, she considers their taste in movies. Thus, given the choice of two men, she prefers Ron to Sam, Sam to Tom, and Tom to Ron. When all three call, she goes out with Tom.

I can't think of a reason in the world for calling her irrational. In fact, I feel she has given the matter more thought then anyone else I know. If one derives their preference relation from multiple considerations, a nontransitive preference relation may result even if the basic considerations are themselves transitive. Of course this is just an instance of the Arrow voting paradox.

Many, if not most, of our preferences will be transitive and, therefore, have the other equivalent properties. But one must be careful before assuming that all of them are or that it is irrational to hold intransitive thoughts. Making choices can be a highly idiosyncratic activity.

A Proof of Theorem 10

The proof strategy for Theorem 10 is summarized in Figure 1. First, stability will be shown to be a direct consequence of transitivity, then it will be shown

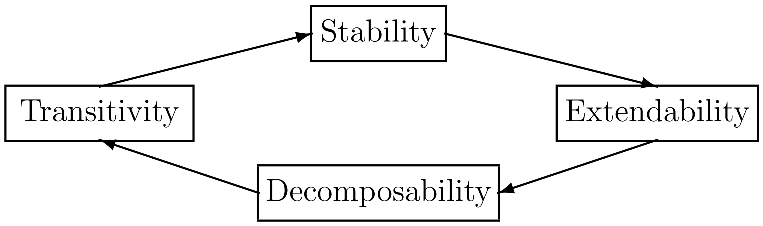

Figure 1: Strategy to prove theorem 10.

that extendability follows from stability and decomposability follows from extendability. Finally, it is shown that decomposability entails transitivity and, therefore, that the four conditions are equivalent for preferences.

Transitivity implies stability: Transitivity implies an order $u_1, \ldots, u_n$ of $\mathcal{U}$ such that $P(T)$ is the least element in T vis-à-vis that order for any nonempty $T \subset \mathcal{U}$. If $S \subset T$ and $P(T) \in S$, then $P(T)$ is the least element in S too. Therefore, $P(S) = P(T)$.

Stability implies extendability: Since $S \subset S \cup T$, stability asserts that $P(S \cup T) \in S \rightarrow P(S) = P(S \cup T)$ which is logically equivalent to $P(S) = P(S \cup T)$ or $P(S \cup T) \notin S$. Therefore, P is extendable.

Extendability implies Decomposability: Extendability entails that both $P(S \cup T) = P(S)$ or $P(S \cup T) \notin S$ and $P(S \cup T) = P(T)$ or $P(S \cup T) \notin T$. Since each disjunction is true, there are two possibilities: (1) $P(S \cup T) = P(S)$ or $P(S \cup T) = P(T)$ or (2) $P(S \cup T) \notin S$ and $P(S \cup T) \notin T$. The first possibility is decomposability and the second is impossible because it entails that $P(S \cup T) \notin S \cup T$, but P is a preference.

Decomposability implies transitivity: Let $u_1 = P(\mathcal{U})$, $u_2 = P(\mathcal{U} \backslash \{u_1\})$, $\ldots$ be an ordering of $\mathcal{U}$ induced by the decomposable preference P. Select an arbitrary $S \subset \mathcal{U}$ and let u_i be the least element in S according to that order. If $P(S) = u_i$, then P is transitive by definition. Let $U = \{u_i, \ldots, u_n\}$, where $n = |\mathcal{U}|$. If $S = U$, then $P(S) = P(U) = u_i$ by construction. Therefore, assume there is a nonempty T such that $T \cup S = U$ and $T \cap S = \emptyset$. By decomposability, either $P(U) = P(S) = u_i$ or $P(U) = P(T)$ but $u_i \notin T$. Therefore, $P(S) = u_i$ and P is transitive. ∎