Continuous Gold Mining

Jeffrey A. Barnett

jbb@notatt.com

Define $P(x)$ to be the probability that gold will be discovered if effort x is committed to excavating a site. Then $P(0) = 0$, P is monotonic nondecreasing, and $P^\infty = \lim_{x\to\infty} P(x) \leq 1$ is the probability that the site contains gold.

Define $P(\Delta x|x)$ to be the probability that committing Δx additional effort will find gold at the site given that effort x did not. Then,

$$P(\Delta x|x) \;\; = \;\; \frac{P(x+\Delta x) - P(x)}{Q(x)},$$

where $Q(x) = 1 - P(x)$. Similarly, define

$$\begin{aligned} Q(\Delta x|x) &= 1 - P(\Delta x|x) \\ &= Q(x+\Delta x)/Q(x) \end{aligned}$$

as the probability that gold will not be found at the site by committing Δx additional effort given that it wasn't found with effort x.

Assume that effort x is spent at the site and that no gold is found. Further assume that we are willing to spend up to $\Delta x = nr$ in additional resources. If gold is found during the excavation, less will be spent because digging stops as soon as treasure is discovered. What is the expected cost of the decision to commit these resources?

This question is straightforward to answer if the process is considered in n discrete steps. Clearly, r is spent for the first step. The second costs the same if it is necessary, i.e., if the first step does not discover gold. Therefore, the contribution of the second step to the total expected cost is $Q(r|x)r$. Similarly, the contribution of the k'th step is $Q\left((k-1)r|x\right)r$. Thus,

$$C(nr|x) \;\; \approx \;\; \sum_{k=1}^{n} Q\left((k-1)r|x\right)r,$$

where $C(y|x)$ is the expected cost when the expenditure of up to y additional resources is committed given that the expenditure of x has not found a treasure. This formula is exact when the summation is replaced by integration, in the limit, as r gets small and nr remains constant, i.e.,

$$\begin{aligned} C(nr|x) &= \int_0^{nr} Q(z|x)\,dz \\ &= \int_0^{nr} Q(x+z)\,dz \Big/ Q(x) \end{aligned}$$

and, recalling that $\Delta x = nr$,

$$C(\Delta x|x) = I(\Delta x|x)/Q(x),$$

where $I(\Delta x|x) = \int_0^{\Delta x} Q(x+z)\,dz$. Let C^∞ be the expected cost to dig at a site until a treasure is found. Then, $C^\infty = C(\infty|0) = \int_0^\infty Q(x)$ since $Q(0) = 1$. Note, $C^\infty < \infty$ implies that $P^\infty = 1$ but the converse may not be true.

Let $\phi(\Delta x|x)$ to be the benefit/cost ratio for committing to effort Δx given that effort x is not productive, i.e.,

$$\begin{aligned} \phi(\Delta x|x) &= P(\Delta x|x)/C(\Delta x|x) \\ &= \frac{P(x+\Delta x) - P(x)}{I(\Delta x|x)}. \end{aligned} \tag{1}$$

The instantaneous benefit/cost ratio, $\phi(x)$, is defined as

$$\begin{aligned} \phi(x) &= \lim_{\epsilon\to 0} \phi(\epsilon|x) \\ &= \lim_{\epsilon\to 0} \frac{P(x+\epsilon) - P(x)}{\int_0^\epsilon Q(x+z)\,dz} \\ &= P'(x)/Q(x), \end{aligned}$$

where the last derivation uses L'Hôspital's rule. Since $P' = -Q'$, it follows that $\phi(x) = -d\log Q(x)/dx$ and, therefore, that $\int \phi = -\log Q$. Thus, for an arbitrary nonnegative function ϕ with the domain $\Re^+$, a corresponding Q and P can be calculated by

$$\begin{aligned} Q(x) &= \exp\left(-\int_0^x \phi(z)\,dz\right) \\ P(x) &= 1 - \exp\left(-\int_0^x \phi(z)\,dz\right). \end{aligned}$$

Since $dI(\Delta x|x)/dx = -(P(x+\Delta x) - P(x))$, a similar line of reasoning and formula 1 show that, for an arbitrary nonnegative $\phi(\Delta x|x)$,

$$I(\Delta x|x) = \exp\left(-\int_0^x \phi(\Delta x|z)\,dz\right).$$

In the case of discrete mining, maximal indivisible blocs are identified by the condition that ϕ of a sequence *decreases* if the sequence is extended in any way. Here, an equivalent *necessary* (but not sufficient) condition is that $d\phi(\Delta x|x)/d\Delta x$ vanish. This derivative is calculated by

$$\frac{d\phi(\Delta x|x)}{d\Delta x} = \frac{P'(x+\Delta x)I(\Delta x|x) - Q(x+\Delta x)\left(P(x+\Delta x) - P(x)\right)}{I(\Delta x|x)^2}.$$

Therefore, the condition that it vanish expands to

$$\begin{aligned}
\frac{d\phi(\Delta x|x)}{d\Delta x} &= 0 \\
P'(x+\Delta x)I(\Delta x|x) &= Q(x+\Delta x)\left(P(x+\Delta x) - P(x)\right) \\
\frac{P'(x+\Delta x)}{Q(x+\Delta x)} &= \frac{P(x+\Delta x) - P(x)}{I(\Delta x|x)} \\
\phi(x+\Delta x) &= \phi(\Delta x|x). \qquad (2)
\end{aligned}$$

It is easy to see, by analogy to the discrete case, that the instantaneous value of ϕ must decrease near the end of a maximal bloc; otherwise extending the bloc increases its ϕ. Thus, another necessary condition that a maximal bloc must satisfy is that, at its end boundary, e

$$\begin{aligned}
\frac{d\phi(e)}{de} &\leq 0 \\
\frac{P''(e)Q(e) - Q'(e)P'(e)}{Q(e)^2} &\leq 0 \\
-Q'(e)P'(e)/Q(e)^2 &\leq -P''(e)/Q(e) \\
P'(e)P'(e)/Q(e)^2 &\leq -P''(e)/Q(e) \\
\phi(e)^2 &\leq -P''(e)/Q(e)
\end{aligned}$$

Therefore, just those e that satisfy this condition can end maximal blocs. Continuity constraints may make it straightforward to find regions of $\Re^+$ where it holds.

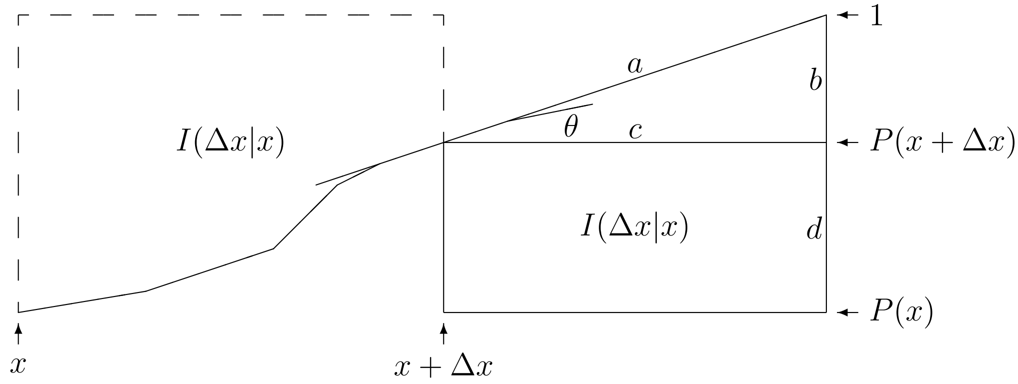

Figure 1: Interpretation of bloc end condition: $\tan(\theta) = P'(x + \Delta x)$.

Figure 1 permits a geometric interpretation of the first end of bloc condition. The curve is $P(z)$ and the line a is the tangent at the point $z = x+\Delta x$. Therefore, its slope is $P'(x + \Delta x) = \ell(b)/\ell(c)$. Since the construction makes $\ell(b) = 1 - P(x+\Delta x) = Q(x+\Delta x)$, it follows that $\ell(c) = Q(x+\Delta x)/P'(x+\Delta x)$ and, therefore, that $\ell(c) = \phi(x + \Delta x)^{-1}$. This is independent of any assumptions about$x + \Delta x$.

Now assume that $x + \Delta x$ ends a maximal bloc. Then formula 2 applies and $\ell(c) = \phi(x + \Delta x)^{-1} = \phi(\Delta x|x)^{-1} = I(\Delta x|x)/(P(x + \Delta x) - P(x))$. By the construction, $\ell(d) = P(x + \Delta x) - P(x)$ and, therefore, the area of the rectangle bound by sides c and d is $\ell(c) \times \ell(d) = I(\Delta x|x)$.

The area bound by the vertical lines at x and $x + \Delta x$, the horizontal line at height 1, and the curve at the bottom is $\int_0^{\Delta x} Q(x + z)\, dz = I(\Delta x|x)$ from the construction and the definition of I. Therefore, the first end of bloc condition states that the area above the curve equals the area of the rectangle.