Dizzy Ants

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Below, a popular puzzle problem is solved along with a few obvious generalization. The original puzzle is the following:

> Four ants are located at the vertices of a square with side length s and each can walk with velocity v. An ant walks toward the ant adjacent to it in the counterclockwise direction. How long is it before they meet and how far has each ant walked?

The generalization supposes n ants each on a vertex of a regular n-gon and the questions are the same. Figure 1 shows the starting conditions for the case

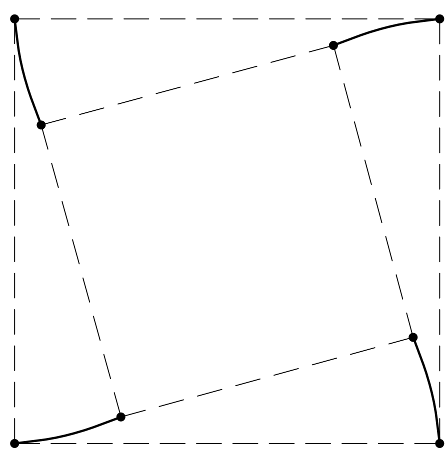
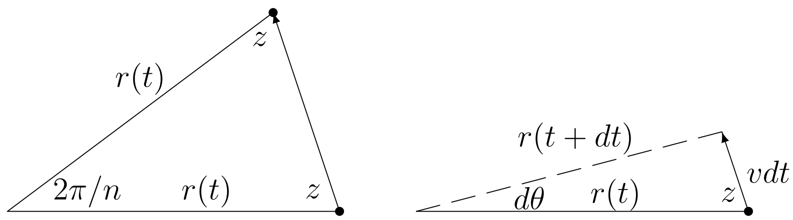


Figure 1: Four ants on a square at times 0 and a short time later.

$n = 4$ and the configuration a short time in the future. Note that the original shape of the convex hull of the ants' position (a square) is maintained but it has shrunk and rotated. This is obvious from considerations of symmetry. Next I derive answers for the n ant problem.

At time $t = 0$ the n ants are equally spaced on a circle of radius r and consecutive ants are spaced by an angle of $2\pi/n$ by radii drawn from the circle's center. Figure 2(a) shows the situation for two ants at time t: the



(a) Situation at time t. (b) Situation at time $t + dt$.

Figure 2: Positions at time t and one ant's progress at $t + dt$.

walk line makes angle $z = \frac{n-2}{2n}\pi$ with the radius. Note that $2\pi/n + 2z = \pi$ because the triangle is isosceles. After time dt, the bottom ant has walked distance vdt and its angle has increased by $d\theta$ from the (Figure 2(b)). Of course the other $n - 1$ ants have made similar, symmetric walks during the same period of time. We will now calculate $r(t)$ and $\theta(t)$ for each ant.

From elementary trigonometry we know $c = \sqrt{a^2 + b^2 - 2ab\cos(C)}$ for any triangle. Thus,

$$r(t + dt) = \sqrt{r(t)^2 + (vdt)^2 - 2r(t)(vdt)cos(z)}.$$

Further,

$$\left.\frac{df(x)}{dx}\right|^{x} = \left.\frac{df(x+w)}{dw}\right|^{w=0}. \tag{1}$$

So if $f = r$, $x = t$, and $w = dt$, then

$$\frac{dr(t)}{dt} = -v\cos(z),$$

or

$$r(t) = r_0 - v\cos(z)t. \tag{2}$$

Now drop a perpendicular from the end of the ant's walk at time dt to the radius and observe that

$$\tan(d\theta) = \frac{v\sin(z)dt}{r(t) - v\cos(z)dt}$$
$$d\theta = \arctan\left(\frac{v\sin(z)dt}{r(t) - v\cos(z)dt}\right)$$

so

$$\theta(t+dt) = \arctan\left(\frac{v\sin(z)dt}{r(t) - v\cos(z)dt}\right) + \theta(t).$$

Now use (1) to show that

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{v\sin(z)}{r(t)} \\ &= \frac{v\sin(z)}{r_0 - v\cos(z)t}\end{aligned} \tag{3}$$

and, therefore, that

$$\theta(t) = \theta_0 + \tan(z)\log\left(\frac{r_0}{r_0 - v\cos(z)t}\right). \tag{4}$$

Solved the above for t as a function of θ and substituted in (2) to show

$$r(\theta) = r_0 e^{(\theta_0 - \theta)\cot(z)}, \tag{5}$$

the motion of an ant, starting at angle θ_0, in polar coordinates. Since

$$\begin{aligned}\cos(z) &= \cos\left(\frac{n-2}{2n}\pi\right) = \sin(\pi/n) \\ \sin(z) &= \sin\left(\frac{n-2}{2n}\pi\right) = \cos(\pi/n),\end{aligned}$$

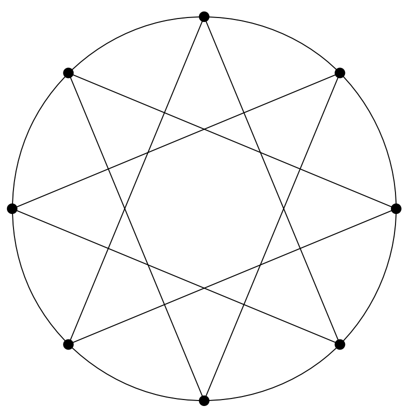
(2), (4), and (5) can be rewritten as

$$\begin{aligned}r(t) &= r_0 - v\sin(\pi/n)t \\ \theta(t) &= \theta_0 + \cot(\pi/n)\log\left(\frac{r_0}{r_0 - v\sin(\pi/n)t}\right) \\ r(\theta) &= r_0 e^{(\theta_0 - \theta)\tan(\pi/n)}\end{aligned}$$

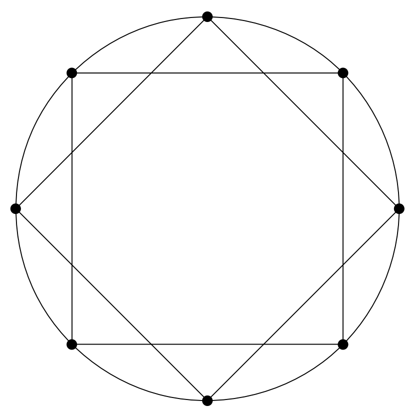
Observe that $r(t) = 0$ when $t = \frac{r_0}{v\sin(\pi/n)}$ so that is the time when the ants meet at the center of the circle. Since an ant walks with velocity v, it moves total distance $r_0/\sin(\pi/n)$.

The above derivations are based on the facts that (1) all ants are equidistant from a point called the center and (2) the angles between chaser/chased pairs are equal. In other words, the derived formulas apply to more general configurations of ants. Assume as above that we have $n > 2$ ants evenly spaced on the circumference of a circle. However, let each ant chase the m'th ant ahead of it in the counterclockwise direction, where $0 < m < n$. Thus, the angular separation between chaser/chased pairs is $2m\pi/n$. In the original problem, $m = 1$. Therefore, the above formulas properly specify ant motions when n/m is substituted everywhere for n.

If $(m, n) = 1$, i.e., m and n are relatively prime integers, then the walk vectors from each ant to the one it follows form a connected path. For example, if $n = 8$ and $m = 3$, the paths form an 8-point star as shown in Figure 3(a). On the other hand, multiple connected paths are formed as



(a) $n = 8$ and $m = 3$.



(b) $n = 8$ and $m = 2$

Figure 3: Examples of generalized walks.

shown in Figure 3(b) where $n = 8$ and $m = 2$. There are two squares and the four ants in one square do not interact with the four ants in the other one. In general the number of separate paths is $P = (m, n)$ and each one involves n/P ants.

Finally assume that the angle separating a chaser/chased pair is $2\alpha\pi$, where $0 < \alpha < 1$ and α is irrational. In this case, the situation modelled will produce a single connected path from all the walk vectors. However, the number of ants will be infinite and the ant locations will be dense in the circumference of the circle of radius $r(t)$ at time t.